Exam 2A Solutions

Curve

If x is your raw grade and y is the curved grade, then

$$y = \begin{cases} 0.8x + 36 & \text{if } x \geq 70 \\ 0.6x + 50 & \text{if } 35 < x < 70 \\ 2x + 1 & \text{if } x \leq 35 \end{cases}$$

Problem 1: Card Probabilities

1A. Probability of exactly 2 heart cards in a 5-card hand

- There are 13 hearts and 39 non-hearts in a standard 52-card deck.
- You must choose 2 hearts from the 13 available and 3 non-hearts from the 39 available.
- **Answer:** $\dfrac{C(13,2) \times C(39,3)}{C(52,5)}$

1B. Probability of at least 2 heart cards in a 5-card hand

- It is easier to find the complement: 1 minus the probability of getting 0 or 1 heart card.
- Ways to get 0 hearts: $C(39,5)$.
- Ways to get 1 heart: $C(13,1) \times C(39,4)$.
- **Answer:** $1 - \dfrac{C(39,5) + C(13,1) \times C(39,4)}{C(52,5)}$

1C. Probability of exactly 2 heart cards and 2 black cards

- A 5-card hand requires one more card. Since it cannot be a heart or black card, it must be a diamond (13 in the deck).
- Choose 2 hearts, 2 black cards (from 26), and 1 diamond.
- **Answer:** $\dfrac{C(13,2) \times C(26,2) \times C(13,1)}{C(52,5)}$

1D. Probability of exactly 2 heart cards OR 2 black cards

- Use the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Probability of 2 hearts is $P(A) = \dfrac{C(13,2) \times C(39,3)}{C(52,5)}$.
- Probability of 2 black cards: $P(B) = \dfrac{C(26,2) \times C(26,3)}{C(52,5)}$.
- Probability of both (from 1C): $P(A \cap B) = \dfrac{C(13,2) \times C(26,2) \times C(13,1)}{C(52,5)}$.
- **Answer:** $\dfrac{C(13,2) \times C(39,3) + C(26,2) \times C(26,3) - C(13,2) \times C(26,2) \times C(13,1)}{C(52,5)}$

Problem 2: Dice

2A. Probability the first four rolls have no 3's, and the fifth roll is a 3

- The probability of not rolling a 3 on a fair dice is 5/6, and rolling a 3 is 1/6.

- **Answer:** $(5/6)^4 \times (1/6)$

2B. Probability that exactly two of five rolls come up as 1's or 2's

- The probability of rolling a 1 or 2 is 2/6, which simplifies to 1/3. The probability of not rolling a 1 or 2 is 2/3.
- We can rephrase this problem as flipping a biased coin with probability 1/3 of heads and probability 2/3 of tails, and we want to find the probability that this coin comes up as heads 2 of the 5 times.
- **Answer:** $C(5,2) \times (1/3)^2 \times (2/3)^3$

Problem 3: Router Connections (~Lesson 11 Problems 1 and 3)

3A. Probability that no router connects

- The routers have success probabilities $p_1 = 0.3$ and $p_2 = 0.2$.
- Failure probabilities are 0.7 and 0.8, respectively.
- **Answer:** $0.7 \times 0.8 = 0.56$

3B. Probability that at least one router connects

- This is the complement of no routers connecting.
- **Answer:** $1 - 0.56 = 0.44$

Problem 4: Multiple-Choice Test

Probability of getting at least 2 questions correct

- There are 10 questions. The student guesses randomly from A, B, C, D, so the success probability is 1/4.
- Find the complement: 1 minus the probability of getting exactly 0 or 1 correct.
- Probability of 0 correct: $(3/4)^{10}$.
- Probability of 1 correct: $C(10,1) \times (1/4)^1 \times (3/4)^9$.
- **Answer:** $1 - \left((3/4)^{10} + 10 \times (1/4) \times (3/4)^9\right)$

Problem 5: Conditional Dice and Coins (~Lesson 9 Problem 2; ~Lesson 12 Problem 3)

5A. Probability the sum is ≤ 4, given the first roll is a 2

- If the first roll is a 2, the six possible outcomes are (2,1), (2,2), (2,3), (2,4), (2,5), (2,6).
- Only 2 outcomes satisfy the condition that the sum is ≤ 4: (2,1) and (2,2).
- **Answer:** $2/6 = 1/3$

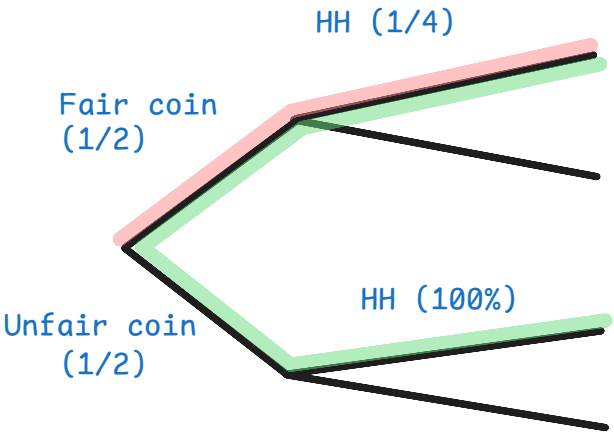
5B. Probability the first roll is a 2, given the sum is ≤ 4

- There are 6 total ways to roll a sum ≤ 4: (1,1), (1,2), (1,3), (2,1), (2,2), (3,1).
- Of those 6 ways, exactly 2 start with a 2: (2,1) and (2,2).
- **Answer:** $2/6 = 1/3$

5C. Probability you were flipping the fair coin given both flips were heads.

We use Bayes' rule: the red branch's probability is divided by the sum of the green branches' probabilities:

Answer: $\dfrac{1/2 \times 1/4}{1/2 \times 1/4 + 1/2 \times 1} = 1/5$



Problem 6 (~Lesson 8 Problem 5)

Typo: In the problem statement, "80 either study math or history" should be changed to "80 either study math or CS".

6A. The student studies math and CS

$$\begin{aligned} P(\text{Math or CS}) &= P(\text{Math}) + P(\text{CS}) - P(\text{Math and CS}) \\ \frac{80}{100} &= \frac{50}{100} + \frac{70}{100} - P(\text{Math and CS}) \\ \Rightarrow P(\text{Math and CS}) &= \frac{40}{100} \end{aligned}$$

6B. The student studies neither subject

This is the complement of "the student studies either math or CS" so the answer is

$$1 - P(\text{Math or CS}) = 1 - \frac{80}{100} = \frac{20}{100}$$

6C. The student studies CS but not math

These are the students who study CS or math, minus those who study math:

$$P(\text{CS or Math}) - P(\text{Math}) = \frac{80}{100} - \frac{50}{100} = \frac{30}{100}.$$

6D. The student studies math but not CS

These are the students who study CS or math, minus those who study CS:

$$P(\text{CS or Math}) - P(\text{CS}) = \frac{80}{100} - \frac{70}{100} = \frac{10}{100}.$$

Problem 7: Netflix User Segmentation (~Lesson 9 Exercise 1)

7A. Probability person is male, given they are 0-19

- **Answer:** $40/(40+30) = 4/7$

7B. Probability person is a 0-19, given they are male

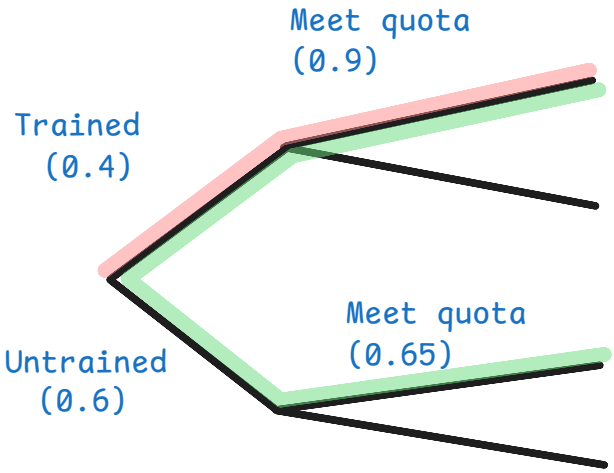
- **Answer:** $40/(40+30+20) = 4/9$

7C. Probability person is not 30+, given they are female

- Females not 30+ includes the 0-19 group (30) and 20-29 group (40), totaling 70. There are 92 females total.
- **Answer:** $(30+40)/(30+40+22)) = 70/92$

Problem 8: Operator Training (~Lesson 12 Problem 1)

Given they meet the quota, probability they attended training



The desired probability is $\dfrac{0.90 \times 0.40}{0.90 \times 0.40 + 0.65 \times 0.60}$ by using Bayes rule on the tree above.

Problem 9: CPU Testing (~Lesson 11 Problem 2)

9A. Probability CPU was tested AND failed some test

- The probability of being tested is 1/3.
- The probability of passing all independent tasks is $(1-0.01) \times (1-0.03) \times (1-0.02)$.
- The probability of failing some test is 1 minus the probability of passing them all.
- **Answer:** $(1/3) \times [1 - (0.99 \times 0.97 \times 0.98)]$

9C. Given a CPU was tested, the probability it failed tasks 2 or 3 is:

$$\begin{aligned} P(\text{fail 2nd or 3rd}) &= P(\text{fail 2nd}) + P(\text{fail 3rd}) - P(\text{fail 2nd and 3rd}) \\ &= P(\text{fail 2nd}) + P(\text{fail 3rd}) - P(\text{fail 2nd})P(\text{fail 3rd}) \\ &= 0.03 + 0.02 - (0.03 \times 0.02) \end{aligned}$$

Here the middle equality is justified since test 2 and test 3 are independent.

Problem 10: Five Dice Rolls

Probability the largest number rolled is a 4

- For the largest number to be exactly 4, all rolls must be ≤ 4, but they cannot *all* be ≤ 3.
- Probability all 5 rolls are ≤ 4 is $(4/6)^5$.
- Probability all 5 rolls are ≤ 3 is $(3/6)^5$.
- **Answer:** $(4/6)^5 - (3/6)^5$

Problem 11

A family has two children. One is a boy born on a Tuesday. What is the probability that both children are boys?

- This is a conditional probability problem where we must count the desired event (both children are boys given that one child is a boy born on a Tuesday) out of the reduced sample space (two children, one of which is a boy born on a Tuesday).
- A child can be a boy/girl born on one of 7 days, giving 14 possible outcomes per child. For two children, there are $14 \times 14 = 196$ total outcomes. We won't need all these possibilities though.
- Outcomes where at least one child is a Tuesday Boy: 14 (child 1) + 14 (child 2) - 1 (both are Tuesday boys) = 27 total combinations.
- Outcomes where *both* are boys, and at least one is a Tuesday Boy: 7 (child 1 is Tuesday Boy, child 2 is any boy) + 7 (child 2 is Tuesday Boy, child 1 is any boy) - 1 (both are Tuesday Boys) = 13 combinations.
- **Answer:** $13/27$